# NOTIZEN

## Total Energy of a Wave in Plasma

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The total mode energy in a plasma is derived from the expression of the energy to second order. This mode energy agrees with that usually given in the literature [1] for wave packets.

Let us consider a homogeneous plasma in an external magnetic field $\boldsymbol{B}_0 = B_{0z}\, \boldsymbol{i}_z$. As a special case we choose the isotropic distribution function at equilibrium: $f_{0\mathrm{q}} = f_{0\mathrm{q}}(v^2)$, $\mathrm{q} = \mathrm{e}, \mathrm{i}$.
Combination of the first two Maxwell equations yields:

$$\frac{1}{8\pi}\frac{\partial}{\partial t}(E^2+B^2) + \boldsymbol{j}\cdot\boldsymbol{E} + \operatorname{div}\frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B}) = 0$$

where $\boldsymbol{E}$, $\boldsymbol{B}$ are the perturbation fields (assuming $\mu = 1$). Assuming for $E$, $B$ a spatial dependence $\sim \exp(i\,\boldsymbol{k}\cdot\boldsymbol{r})$ we obtain after integrating over a spatial period:

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint \mathrm{d}r\left(\frac{|E|^2+|B|^2}{8\pi}\right) + \oint \mathrm{d}r\left(\frac{\boldsymbol{j}\cdot\boldsymbol{E}^* + \boldsymbol{j}^*\cdot\boldsymbol{E}}{2}\right) = 0\,.$$

We now assume spatial integration over one period and discard the sign $\oint \mathrm{d}r$.

From the linearized Vlasov equation, we have

$$\frac{\mathrm{d}f_{1\mathrm{q}}}{\mathrm{d}t} = -\frac{2\,e_\mathrm{q}}{m_\mathrm{q}}\,\boldsymbol{v}\cdot\boldsymbol{E}\,\frac{\mathrm{d}f_{0\mathrm{q}}}{\mathrm{d}v^2} \qquad \text{so that}$$

$$\boldsymbol{j}\cdot\boldsymbol{E}^* = \sum_{\mathrm{e,i}} e_\mathrm{q}\int \mathrm{d}^3v\, f_{1\mathrm{q}}\,\boldsymbol{v}\cdot\boldsymbol{E}^* + O(\lambda^3\ldots)\,, \quad \lambda \cong \left|\frac{f_{1\mathrm{q}}}{f_{0\mathrm{q}}}\right|$$

$$= -\sum_{\mathrm{e,i}}\frac{m_\mathrm{q}}{2}\int \mathrm{d}^3v\, f_{1\mathrm{q}}\,\frac{\mathrm{d}f^*_{1\mathrm{q}}/\mathrm{d}t}{\mathrm{d}f_{0\mathrm{q}}/\mathrm{d}v^2} + O(\lambda^3\ldots)$$

and

$$\mathrm{d}W/\mathrm{d}t + O(\lambda^3\ldots) = 0$$

where

$$W = \frac{|E|^2+|B|^2}{8\pi} - \sum_{\mathrm{e,i}}\frac{m_\mathrm{q}}{4}\int \mathrm{d}^3v\,\frac{|f_{1\mathrm{q}}|^2}{\mathrm{d}f_{0\mathrm{q}}/\mathrm{d}v^2}\,. \qquad (1)$$

For a wave packet, the total energy density is given by the formula

$$W' = \frac{1}{8\pi}\left[E_i^*\,\frac{\partial}{\partial\omega_\mathrm{r}}\,(\omega_\mathrm{r}\,\varepsilon'_{ij})\,E_j + |B|^2\right] \qquad (2\,\mathrm{a})$$

$$= \frac{1}{8\pi\,\omega_\mathrm{r}}\left[E_i^*\,\frac{\partial}{\partial\omega_\mathrm{r}}\,(\omega_\mathrm{r}^2\,\varepsilon'_{ij})\,E_j\right] \qquad (2\,\mathrm{b}) \qquad (2)$$

where $\varepsilon'_{ij} = \operatorname{Re}\varepsilon_{ij}$ ($\varepsilon_{ij}$ = equivalent dielectric tensor)

$$\omega_\mathrm{r} = \operatorname{Re}\omega\,, \qquad \omega_\mathrm{i} = \operatorname{Im}\omega\,.$$

Such an expression is obtained [1, 2] by distinguishing between slow and fast time dependence

$$E \cong E_0(t)\,\exp(i\,\omega\,t)\,,$$

and the transition from (2 a) to (2 b) follows from the Maxwell equations in the lowest order.

The solution $f_{1\mathrm{q}}$ represents for the macroscopic quantities a pure mode $[\boldsymbol{k},\ \omega(\boldsymbol{k})]$, only in its asymptotic limit $t \to \infty$:

$$f_{1\mathrm{q}} = -\frac{2\,e_\mathrm{q}}{m_\mathrm{q}}\,\frac{\mathrm{d}f_{0\mathrm{q}}}{\mathrm{d}v^2}\int^t \mathrm{d}t'\,E_i(\boldsymbol{r}',t')\,v_i'$$

where integration is made along the unperturbed orbits:

$$\boldsymbol{r}' = \boldsymbol{r} + \frac{v_\perp}{\Omega_\mathrm{q}}\{\sin[\Omega_\mathrm{q}(t'-t)+\vartheta] - \sin\vartheta\}\,\boldsymbol{i}_x + \frac{v_\perp}{\Omega_\mathrm{q}}\{\cos\vartheta - \cos[\Omega_\mathrm{q}(t'-t)+\vartheta]\}\,\boldsymbol{i}_y + v_\parallel(t'-t)\,\boldsymbol{i}_z\,,$$

$$\Omega_\mathrm{q} = e_\mathrm{q}\,B_0/m_\mathrm{q}\,c\,.$$

In this limit $t \to \infty$, we can neglect the free-streaming term $[\sim \exp(i\,\boldsymbol{k}\cdot\boldsymbol{v}\,t)]$, which makes a zero contribution to the velocity space integral.

We get for $f_{1\mathrm{q}}$:

$$f_{1\mathrm{q}} = -\frac{2\,e_\mathrm{q}}{m_\mathrm{q}}\,\frac{\mathrm{d}f_{0\mathrm{q}}}{\mathrm{d}v^2}\,\exp\left\{-\frac{i\,k_\perp\,v_\perp\,\sin\vartheta}{\Omega_\mathrm{q}}\right\}\sum_{n=-\infty}^{n=+\infty}\frac{e^{in\vartheta}[(v_\parallel E_\parallel + n\,\Omega_\mathrm{q}\,E_x/k_\perp)\,J_n - i\,v_\perp\,E_y\,J_n']}{i(\omega + k_\parallel\,v_\parallel + n\,\Omega_\mathrm{q})}$$

[1] L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Pergamon Press, London 1960.

[2] Yu. L. Klimontovich, The Statistical Theory of Nonequilibrium Processes in a Plasma, Pergamon Press, London 1967.

where the components of $E$ are taken at $(\mathbf{r}, t)$ and where $J_n = J_n(k_\perp v_\perp/\Omega_q)$. After $\vartheta$-integration, we get for $W$:

$$W = \left(\frac{|E|^2 + |B|^2}{8\pi}\right) - \sum_{e,i} \frac{e_q^2}{m_q} \int_0^{+\infty} v_\perp \, dv_\perp \int_{-\infty}^{+\infty} dv_\| \frac{df_{0q}}{dv^2} \cdot \sum_{n=-\infty}^{n=+\infty} \frac{|A_n|^2}{(\omega_r + k_\| v_\| + n\,\Omega_q)^2 + \omega_i^2} \tag{1'}$$

with

$$A_n = (v_\| E_\| + n\,\Omega_q E_x/k_\perp) J_n - i\, v_\perp E_y J_n'$$

$|A_n|^2$ has no $\omega$-dependence.

On the other hand, the dielectric tensor is defined by:

$$\varepsilon_{ij} E_j = E_i - \sum_{e,i} \frac{2\,\omega^2}{i\,\omega} \int d^3v\, v_i \int^t dt'\, E_j(\mathbf{r}', t')\, v_j' \frac{df_{0q}}{dv^2},$$

so that

$$E_i^* \varepsilon_{ij} E_j = |E|^2 + \sum_{e,i} \frac{2\,\omega_{pq}^2}{\omega^2} \int d^3v \frac{df_{0q}}{dv^2} \sum_{n=-\infty}^{n=+\infty} \sum_{n'=-\infty}^{n'=+\infty} \frac{E_i^* v_i \exp\{i(n-n')\,\vartheta\} A_n J_{n'}}{\omega + k_\| v_\| + n\,\Omega_q}$$

with $\omega_{pq}$ = plasma frequency of species q and after $\vartheta$-integration:

$$E_i^* \varepsilon_{ij} E_j = |E|^2 + \sum_{e,i} \frac{2\,\omega_{pq}^2}{\omega} \int_0^{+\infty} v_\perp \, dv_\perp \int_{-\infty}^{+\infty} dv_\| \frac{df_{0q}}{dv^2} \sum_{n=-\infty}^{n=+\infty} \frac{|A_n|^2}{\omega + k_\| v_\| + n\,\Omega_q}.$$

Assuming $|\omega_i/\omega_r| \ll 1$ and $|\mathrm{Im}\,\varepsilon_{ij}| \ll 1$ we have:

$$E_i^* \cdot \mathrm{Re}(\omega\,\varepsilon_{ij}) \cdot E_j \cong E_i^* (\omega_r\,\varepsilon_{ij}')\, E_j$$

and

$$\frac{1}{8\pi} E_i^* \frac{\partial}{\partial \omega_r} (\omega_r\,\varepsilon_{ij}')\, E_j \cong \frac{|E|^2}{8\pi} - \sum_{e,i} \frac{e_q^2}{m_q} \int_0^{+\infty} v_\perp \, dv_\perp \int_{-\infty}^{+\infty} dv_\| \frac{df_{0q}}{dv^2} \sum_{n=-\infty}^{n=+\infty} \frac{|A_n|^2}{\omega_r + k_\| v_\| + n\,\Omega_q}$$

and

$$W' - W = \Delta W \sim (\omega_i/\omega_r)^2\, W.$$

This result could probably be generalized for arbitrary distribution functions and slightly inhomogeneous plasmas. It shows clearly that the wave packet energy [1, 2] is approximately the same as the total energy of one wave derived from the expression of the energy to second order. This energy should be relevant for non-linear mode coupling because it is the total energy of a mode.

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